Multivariable Calculus

Quiz 12 **SOLUTIONS**

1) Verify that the vector field $\boldsymbol{F}$ below is conservative. Then find a potential function f so that $\boldsymbol{F} = \boldsymbol{\nabla} f$.

$$\boldsymbol{F}(x, y, z) = \left\langle 3x^2y^2z,\ 2x^3yz - 2y + \sin(z),\ x^3y^2 + y\cos(z) \right\rangle$$

Solution: We first check that this vector field is conservative.

$$\begin{aligned} P_y &= 6x^2yz = Q_x \qquad \checkmark \\ P_z &= 3x^2y^2 = R_x \qquad \checkmark \\ Q_z &= 2x^3y + \cos(z) = R_y \qquad \checkmark \end{aligned}$$

To find the potential function, we can begin with

$$f(x, y, z) = \int 3x^2y^2z\ dx = x^3y^2z + g(y, z).$$

Taking the y-partial gives us

$$\begin{aligned} f_y(x, y, z) &= 2x^3yz + g_y(y, z) = 2x^3yz - 2y + \sin(z), \\ g_y(y, z) &= -2y + \sin(z), \\ g(y, z) &= -y^2 + y\sin(z) + h(z). \end{aligned}$$

which means

$$f(x, y, z) = x^3y^2z - y^2 + y\sin(z) + h(z).$$

Finally,

$$\begin{aligned} f_z(x, y, z) &= x^3y^2 + y\cos(z) + h'(z) = x^3y^2 + y\cos(z), \\ h'(z) &= 0, \\ h(z) &= C. \end{aligned}$$

So, the family of potential functions for $\boldsymbol{F}$ is given by

$$f(x, y, z) = x^3y^2z - y^2 + y\sin(z) + C.$$

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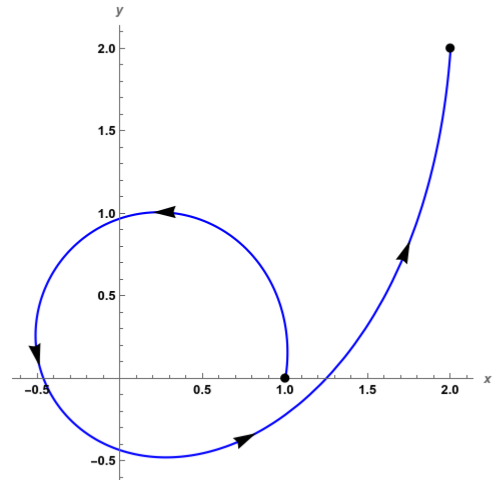
2) Compute the line integral

$$\int_{\mathcal{C}} \left\langle \ln\left(y^2+1\right) - 2x,\ \frac{2xy}{y^2+1} \right\rangle \cdot d\vec{r}$$

where $\mathcal{C}$ is the curve shown below (which begins at $(1,0)$ and ends at $(2,2)$).



Solution: Note that the vector field is conservative!

$$P_y = \frac{2y}{y^2+1} = Q_x \qquad \checkmark$$

This means the line integral above only depends on the two endpoints. To find the potential function, we compute

$$f(x,y) = \int \left(\ln\left(y^2+1\right) - 2x\right)\, dx = x\ln(y^2+1) - x^2 + g(y).$$

Since

$$f_y(x,y) = \frac{2xy}{y^2+1} + g'(y) = \frac{2xy}{y^2+1}$$

we can simply take $g(y) = 0$. This gives us the potential function

$$f(x,y) = x\ln(y^2+1) - x^2.$$

Finally,

$$\begin{aligned}\int_{\mathcal{C}} \left\langle \ln\left(y^2+1\right) - 2x,\ \frac{2xy}{y^2+1} \right\rangle \cdot d\vec{r} &= f(2,2) - f(1,0) \\ &= (2\ln(5) - 4) - (1\ln(1) - 1) \\ &= 2\ln(5) - 3\end{aligned}$$